United Kingdom Mathematics Trust

Intermediate Mathematical Olympiad

Cayley paper

Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. A four-digit number, n, is written as '$ABCD$' where A, B, C and D are all different odd digits. It is divisible by each of A, B, C and D. Find all the possible numbers for n.

SOLUTION

'$ABCD$' is a number composed of four out of five digits 1, 3, 5, 7, 9. Consider two cases:

Case 1. 9 is not one of the digits A, B, C, D.

Number '$ABCD$' is a multiple of 3, so $A+B+C+D$ is also a multiple of 3. But $A+B+C+D = 1+3+5+7 = 16$, which is not a multiple of 3. This is a contradiction, so there are no solutions in this case.

Case 2. 9 is one of the digits A, B, C, D.

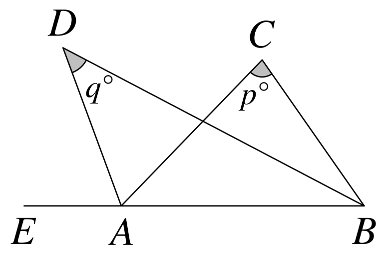
Number '$ABCD$' is a multiple of 9, so $A + B + C + D$ is also a multiple of 9.

Let x be the digit **not** used in '$ABCD$'. Then $A + B + C + D = 1 + 3 + 5 + 7 + 9 - x = 25 - x$ and this must be a multiple of 9. Hence $x = 7$.

This means 1, 3, 5, 9 are the digits used, so '$ABCD$' is a multiple of 5. Hence $D = 5$. We can rearrange 1, 3, 9 as A, B, C in six different ways.

All the possible numbers for n are 1395, 1935, 3195, 3915, 9135, 9315. All of them are divisible by 1, 3 and 9 (as their sum of digits is always 18), and also by 5 (as they all end in a 5).

2. The diagram shows a triangle ABC with side BA extended to a point E. The bisector of $\angle ABC$ meets the bisector of angle $\angle EAC$ at D. Let $\angle BCA = p^\circ$ and $\angle BDA = q^\circ$.
Prove that $p = 2q$.



SOLUTION

Let $a^\circ = \angle EAD = \angle DAC$ and $b^\circ = \angle ABD = \angle DBC$.

Points E, A, B are collinear, so $\angle CAB = (180 - 2a)^\circ$.

Adding up the angles in triangle ADB:

$$a + (180 - 2a) + q + b = 180$$

$$\therefore\ q = a - b$$

Similarly, considering the angles in triangle ACB:

$$(180 - 2a) + p + 2b = 180$$

$$\therefore\ p = 2a - 2b = 2q.$$

3. Aroon's PIN has four digits. When the first digit (reading from the left) is moved to the end of the PIN, the resulting integer is 6 less than 3 times Aroon's PIN. What could Aroon's PIN be?

SOLUTION

Let's denote Aroon's PIN by $1000a + b$, where $0 \leq a \leq 9$ is its first digit and $0 \leq b < 1000$ is the three-digit number formed by the remaining digits of his PIN.

When the first digit of Aroon's PIN is moved to the end, the new number is $10b + a$. Hence:

$$3(1000a + b) - 6 = 10b + a$$

$$\therefore 2999a = 7b + 6$$

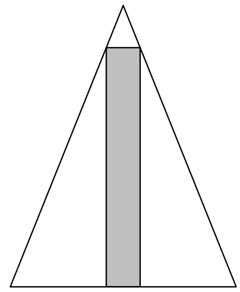
Given $0 \leq b < 1000$, we have $6 \leq RHS < 7006$, hence $0 < a < 3$, so a must be 1 or 2.

If $a = 1$, we have $7b + 6 = 2999$, so $b = 427\frac{4}{7}$, which is not a whole number.

If $a = 2$, we have $7b + 6 = 5998$, so $b = 856$. We can easily check that $2856 \times 3 - 6 = 8562$.

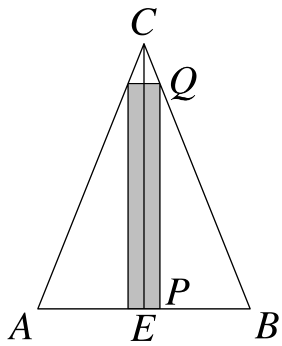
Hence Aroon's PIN must be 2856.

4. The diagram shows a rectangle inside an isosceles triangle. The base of the triangle is n times the base of the rectangle, for some integer n greater than 1. Prove that the rectangle occupies a fraction $\frac{2}{n} - \frac{2}{n^2}$ of the total area.



SOLUTION

Denote the base of the rectangle by $s = \frac{AB}{n}$ and label the points as on the diagram (with E being the midpoint of AB). The ratio between the rectangle area and the triangle area is equal to:



$$R = \frac{s \times PQ}{\frac{1}{2}AB \times EC} = \frac{2s}{AB} \times \frac{PQ}{EC} = \frac{2}{n} \times \frac{PQ}{EC}.$$

Triangles PBQ and EBC are similar (their respective sides are parallel):

$$\frac{PQ}{EC} = \frac{PB}{BE}$$

$$\therefore \frac{PQ}{EC} = \frac{2PB}{2BE} = \frac{AB - s}{AB} = 1 - \frac{1}{n}$$

$$\therefore R = \frac{2}{n}\left(1 - \frac{1}{n}\right) = \frac{2}{n} - \frac{2}{n^2}$$

5. The whole numbers from 1 to $2k$ are split into two equal-sized groups in such a way that any two numbers from the same group share no more than two distinct prime factors. What is the largest possible value of k?

SOLUTION

Numbers $30, 60$ and 90 all share three distinct prime factors (2, 3 and 5). If $k \geq 45$, each of these three numbers must be assigned to one of the two groups, so one of these groups must contain at least two of these numbers. Hence $k < 45$.

We need to check that there is a way of splitting the numbers into two groups when $k = 44$.

If $k = 44$, we can split the numbers into $A = \{1, 2, 3, \ldots, 44\}$ and $B = \{45, 46, \ldots, 87, 88\}$.

We need to show that using this way of splitting the numbers there are not two numbers, x and y, which share three prime factors and are in the same group.

Let $x < y \leq 88$ share three prime factors (p, q, r). Then both x, y are multiples of pqr.
As $3pqr \geq 3 \times (2 \times 3 \times 5) = 90 > 88$, we have $x = pqr$ and $y = 2pqr$.

As $y \leq 88$ and $x = \frac{1}{2}y$, we have $x \leq 44$, so x is in group A.

As $y = 2pqr \geq 2 \times (2 \times 3 \times 5) > 44$, we have y is in group B.

Hence 44 is the largest possible value of k.

6. A bag contains 7 red discs, 8 blue discs and 9 yellow discs. Two discs are drawn at random from the bag. If the discs are the same colour then they are put back into the bag. However, if the discs are different colours then they are removed from the bag and a disc of the third colour is placed in the bag. This procedure is repeated until there is only one disc left in the bag or the only remaining discs in the bag have the same colour. What colour is the last disc (or discs) left in the bag?

SOLUTION

Denote the number of red disks by R, blue disks by B and yellow disks by Y.
We will prove that at any point of the game either:

(a) R, Y are both odd and B is even or

(b) R, Y are both even and B is odd.

At the start, the game is in the state (a) above. If we draw two discs of the same colour, the total number of disks of each colour does not change. If we draw two discs of different colours, each of R, Y, B changes by one, changing the parity of all three of these values, hence moving the game between states (a) and (b).

The game cannot end in state (a) as it means $R, Y > 0$, so it must end in state (b). As in state (b), B cannot be zero, the last disc (or discs) left in the bag is blue.